Combinatorics

Extremal, Probabilistic and Additive

July 23–27, 2018

University of São Paulo, Brazil

Monday, 23 July 2018

10:00–10:40 Imre Leader (Cambridge)

Decomposing the complete r-graph. The Graham–Pollak theorem states that to decompose the complete graph K_n into complete bipartite subgraphs we need at least $n-1$ of them. What happens for hypergraphs? In other words, suppose that we wish to decompose the complete r-graph on n vertices into complete r-partite r-graphs; how many do we need?

In this talk we will report on recent progress on this problem. This is joint work with Luka Milićević and Ta Sheng Tan.

10:40–11:20 Bhargav Narayanan (Rutgers)

Symmetric intersecting families. A family of sets is said to be intersecting if any two sets in the family have nonempty intersection. Families of sets subject to various intersection conditions have been studied over the last fifty years, and a common feature of many of the results in the area is that the extremal families are often quite asymmetric. What changes if we consider intersecting families that are constrained to be symmetric? I will talk about two recent results that shed some light on this question.

Based on joint works with David Ellis, Keith Frankston and Jeff Kahn.

11:30–12:10 Zoltán Füredi (Rényi Institute and UIUC)

Nearly-subadditive sequences. The sequence $a(1)$, $a(2)$, $a(3), \dots$ of reals is called *subadditive* if $a(n+m) \leqslant a(n) + a(m)$ holds for all integers $n, m \geqslant 1$. Fekete's lemma (1923) states that the sequence $\{a(n)/n\}$ has a limit (possibly negative infinity). Let $f(n)$ be a non-negative, non-decreasing sequence. deBruijn and Erdős (1952) called the sequence $\{a(n)\}$ *nearly* f-subadditive if

$$a(n+m) \leqslant a(n) + a(m) + f(n+m) \tag{1}$$

holds for all $n \leqslant m \leqslant 2n$. They showed that if the error term f is small,

$$\sum_{n=1}^{\infty} f(n)/n^2 \quad \text{is finite,} \tag{2}$$

then the limit $\{a(n)/n\}$ still exists. Their result is listed in the Bollobás-Riordan book (2006) as one of the most useful tools in Percolation Theory. Also, recurrence relations of type (1) are often encountered in the analysis of divide and conquer algorithms, see, e.g., Hsien-Kuei Hwang and Tsung-Hsi Tsai (2003).

We show that the deBruijn–Erdős condition (2) for the error term is not only sufficient but also necessary in the following strong sense. If $\sum_{n=1}^{\infty} f(n)/n^2 = \infty$, then there exists a nearly f-subadditive sequence $\{b(n)\}$ such that the sequence of slopes $\{b(n)/n\}$ takes every rational number. On the other hand, we show that their condition can be weakened such that the limit exists if (1) holds only for the pairs $n \leqslant m \leqslant cn$ for some fixed $c > 1$.

Joint works with I. Ruzsa.

References

[1] Béla Bollobás and Oliver Riordan: *Percolation.* Cambridge University Press, New York, 2006. x+323 pp.

[2] N. G. de Bruijn and P. Erdős: Some linear and some quadratic recursion formulas, I., *Indag. Math.* **13** (1951), 374–382.

[3] N. G. de Bruijn and P. Erdős: Some linear and some quadratic recursion formulas, II., *Indag. Math.* **14** (1952), 152–163.

[4] Hsien-Kuei Hwang and Tsung-Hsi Tsai: An asymptotic theory for recurrence relations based on minimization and maximization. *Theoret. Comput. Sci.* **290** (2003), 1475–1501.

14:00–14:40 Julian Sahasrabudhe (IMPA and Cambridge)

Central limit theorems from constrained zeros of probability generating functions. For each n, let $X_n \in \{0, \dots, n\}$ be a random variable with mean μ_n and standard deviation σ_n, and let

$$P_n(z) = \sum_{k=0}^{n} \mathbb{P}(X_n = k) z^k$$

be its probability generating function. We show that if none of the complex zeros of the polynomials $\{P_n(z)\}$ are contained in a neighbourhood of $1 \in \mathbb{C}$ and $\sigma_n > n^{\epsilon}$, for some $\epsilon > 0$, then $X_n^* = (X_n - \mu_n)\sigma_n^{-1}$ tends to a normal random variable $Z \sim N(0,1)$ in distribution as $n \to \infty$. Moreover, we show this result is sharp in the sense that there exist sequences of random variables $\{X_n\}$ with $\sigma_n > C \log n$ for which $P_n(z)$ has no roots near 1 and X_n^* is not asymptotically normal. These results disprove a conjecture of Pemantle and improve upon various results in the literature. We go on to prove several other results connecting the location of the zeros of $P_n(z)$ and the distribution of the random variables X_n. This is joint work with Marcus Michelen.

14:40–15:20 Shoham Letzter (ETH Zürich and Cambridge)

Lagrangians of hypergraphs. Frankl and Füredi conjectured (1989) that any r-uniform hypergraph, whose edges form an initial segment of length m in the colex ordering, maximises the Lagrangian among all r-uniform hypergraphs with m edges, for all r and m. We prove this conjecture for $r = 3$ (and all sufficiently large m), thus improving results of Talbot (2002), Tang et al. (2016) and Tyomkyn (2017). For larger r, we show that the conjecture holds whenever $\binom{t-1}{r} \leqslant m \leqslant \binom{t-1}{r} + \binom{t-2}{r-1}$ for some integer t (and m is large enough), thus improving a result of Tyomkyn. However, the conjecture is in fact false for general $r \geqslant 4$ and m.

We also find an interesting connection between the problem of characterising the maximisers of the Lagrangian and the problem of maximising the sum of degrees squared in r-uniform hypergraphs with prescribed order and size.

This is joint work with Vytautas Gruslys and Natasha Morrison.

16:20–17:00 Ervin Győri (Rényi Institute and Central European University)

C_{2k+1}-free graphs and hypergraphs. In this talk, we investigate graphs and (mostly 3-uniform) hypergraphs not containing any copy of the cycle C_5 or in general the odd cycle C_{2k+1}.

Graphs. We study the maximum number of triangles in C_5-free graphs, first studied by Bollobás and the speaker, improved later by Alon and Shikhelman and now by Ergemlidze, Methuku, Salia and by the speaker. The best present bounds are

$$\frac{1}{3\sqrt{3}}(1+o(1))n^{3/2} \leqslant \mathrm{ex}(n, K_3, C_5) \leqslant \frac{1}{2\sqrt{2}}(1+o(1))n^{3/2}.$$

In C_{2k+1}-free graphs it was studied by Hao Li and the speaker, then it was improved by Füredi and Özhahya, but here the lower and upper bounds are much farther from each other. With Ergemlidze and Methuku, we asymptotically determined the maximum possible number of edges in a C_{2k+1}-free graph containing no induced copy of $K_{s,t}$ if $k \geqslant 2$, $s = 2$ and $t \geqslant 2$ or $s = t = 3$ (except when $k = s = t = 2$ but we improve the estimate in this case too). This strengthens a result of Allen, Keevash, Sudakov and Verstraete and answers a question of Loh, Tait and Timmons.

Hypergraphs. The number of hyperedges in C_5-free 3-uniform hypergraphs first was studied by Bollobás and the speaker too, and it was improved by Füredi and Özhahya. For C_{2k+1}-free hypergraphs, the extremal number was estimated by Lemons and the speaker. Recently, we studied it in 3-uniform linear hypergraphs. Let $\mathbf{F}$ be a family of 3-uniform linear hypergraphs. The linear Turán number of $\mathbf{F}$ is the maximum possible number of edges in a 3-uniform linear hypergraph on n vertices which contains no member of $\mathbf{F}$ as a subhypergraph. We asymptotically determine this Turán number for C_5 (in the Berge sense). With Ergemlidze and Methuku, we also show that the linear Turán number of the four cycle C_4 and $\{C_3, C_4\}$ are equal asymptotically, which is a strengthening of a theorem of Lazebnik and Verstraete. We establish a connection between the linear Turán number of the linear cycle of length $2k+1$ and the extremal number of edges in a graph of girth more than $2k-2$. Combining our result and a theorem of Collier-Cartaino, Graber and Jiang, we obtain that the linear Turán number of the linear cycle of length $2k+1$ is $\Theta(n^{1+1/k})$ for $k = 2$, 3, 4 and 6.

17:00–17:40 Imre Bárány (Rényi Institute and University College London)

Theorems of Carathéodory, Helly, and Tverberg with no dimension. Carathéodory's classic result from 1907 says that if a point a lies in the convex hull of a set $P \subset R^d$, then it lies in the convex hull of a subset $Q \subset P$ of size at most $d+1$. What happens if we want a subset Q of size $r < d+1$ such that $a \in \operatorname{conv} Q$? In general, this is impossible as $\operatorname{conv} Q$ is too low dimensional. We offer some remedy: p is close to $\operatorname{conv} Q$ for some subset Q of size k, in an appropriate sense. Similar results hold for the famous Helly and Tverberg theorems as well. This is joint work with Karim Adiprasito and Nabil Mustafa.

Tuesday, 24 July 2018

10:00–10:40 Benny Sudakov (ETH Zürich)

Rainbow structures, Latin squares & graph decompositions. A subgraph of an edge-coloured graph is called rainbow if all its edges have distinct colours. The study of rainbow subgraphs goes back to the work of Euler on Latin squares. Since then rainbow structures were the focus of extensive research and found applications in design theory and graph decompositions. In this talk we discuss how probabilistic reasoning can be used to attack several old problems in this area. In particular we show that well known conjectures of Ryser, Hahn, Ringel, and Graham-Sloane hold asymptotically.

Based on joint works with Alon, Montgomery, and Pokrovskiy.

10:40–11:20 Liana Yepremyan (Oxford)

On the number of symbols that forces a transversal. Akbari and Alipour conjectured that any Latin array of order n with at least $n^2/2$ symbols contains a transversal, or equivalently, every proper-edge coloring of the complete bipartite graph $K_{n,n}$ with $n^2/2$ colours contains a rainbow perfect matching. In this talk we will present a proof of this conjecture in a stronger sense: we show that $n^{399/200}$ colours suffice. This is joint work with Peter Keevash.

11:30–12:10 Yufei Zhao (MIT)

A reverse Sidorenko inequality. Given a graph parameter $f(G) = \hom(G, H)$, among graphs G with a given degree distribution, which G maximizes $f(G)^{1/v(G)}$? We show that in many situations, e.g., the number of independent sets, the maximizing G is a disjoint union of complete bipartite graphs, thereby proving conjectures of Kahn, Galvin–Tetali, and Cohen–Csikvári–Perkins–Tetali.

Joint work with Ashwin Sah, Mehtaab Sawhney, and David Stoner.

14:00–14:40 Michael Krivelevich (Tel Aviv University)

Cycle and cycle lengths in expanders. A graph G on n vertices is called an α-expander if the external neighborhood of every vertex subset U of size $|U| \leqslant n/2$ in G has size at least $\alpha|U|$.

We discuss cycles and cycle lengths in α-expanders.

A joint work with Rajko Nenadov, ETH Zürich.

14:40–15:20 Natasha Morrison (Cambridge)

Maximising the number of induced cycles in a graph. How many induced cycles can a graph on n vertices contain? For sufficiently large n, we determine the maximum number of induced cycles and the maximum number of even or odd induced cycles. We also characterize the graphs achieving this bound in each case. This answers a question of Tuza, and a conjecture of Chvátal and Tuza from 1988. Joint work with Alex Scott.

16:20–17:00 Maya Stein (University of Chile)

Embedding trees with minimum and maximum degree conditions. We will explore what kind of degree condition needs to be imposed on a graph G so that it contains any tree of a given size as a subgraph. There exist several conjectures in this direction, the most famous being the Erdős–Sós conjecture. We prove a result that involves the maximum and the minimum degree of the host graph. Our result is for for bounded degree trees and dense host graphs, but we conjecture it holds in general. Applications to the Erdős–Sós and to a recent conjecture by Havet, Reed, Stein and Wood are also discussed. This is joint work with G. Besomi and M. Pavez-Signé.

17:00–17:40 Jie Han (University of São Paulo)

Spanning structures in pseudorandom graphs and hypergraphs. The study of random and pseudorandom graphs is a central topic in probabilistic combinatorics. It is known that (large) spectral gaps imply spanning structures, such as, perfect matchings, Hamiltonian cycles and clique-factors. We will survey the existing results and talk about some recent developments. This talk includes joint work with Hiệp Hàn, Yoshiharu Kohayakawa, Patrick Morris and Yury Person.

Wednesday, 25 July 2018

10:00–10:40 David Conlon (Oxford)

Ramsey complete sequences. A sequence of positive integers A is said to be entirely Ramsey complete if, for any two-colouring of A, every positive integer can be written as the sum of distinct elements of A of the same colour. We show that there exists a constant C and an entirely Ramsey complete sequence A such that $|A \cap [n]| \leqslant C \log^2 n$ for all n. This is best possible up to the constant and solves a problem of Burr and Erdős. We also discuss several related problems stated by the same authors.

Joint work with Jacob Fox.

10:40–11:20 Anita Liebenau (Monash University)

Ramsey equivalence in many colours. Two graphs H and F are called q-Ramsey equivalent if any graph G is q-Ramsey for H if and only if it is q-Ramsey for F (where G is q-Ramsey for H, say, if every colouring of the edges of G with q colours contains a monochromatic copy of H). Fox et al. proved that the only connected graph which is 2-Ramsey equivalent to the complete graph K_n is K_n itself. It is an open question whether there exist two connected non-isomorphic graphs that are 2-Ramsey equivalent.

This question may be easier to answer for $q > 2$ colours. If H and F are (not) 2-equivalent then this does not imply a priori that they are (not) q-equivalent for $q > 2$. We address questions on q-Ramsey equivalence for $q > 2$. Using similar methods we prove that every graph G which is minimal 2-Ramsey for K_n is contained in a graph G' which is minimal q-Ramsey for K_n, for all $q > 2$.

Joint with Dennis Clemens and Damian Reding.

11:30–12:10 Simon Griffiths (PUC–Rio)

Chromatic thresholds of graphs. The chromatic threshold $\delta_\chi(H)$ of a graph H is the infimum of $d > 0$ such that there exists $C = C(H, d)$ for which every H-free graph G with minimum degree at least $d|G|$ satisfies $\chi(G) \leqslant C$. In this talk we briefly discuss results in the area leading up to the determination of $\delta_{\chi(H)}$ for all graphs H. In addition we discuss variants of these results in the context of random graphs, and related open problems.

Based on joint work with Peter Allen, Julia Böttcher, Yoshiharu Kohayakawa and Robert Morris.

Thursday, 26 July 2018

10:00–10:40 Ehud Friedgut (Weizmann)

The sharp threshold criterion and the container method; chisel and hammer. I will present a sharp threshold result regarding Ramsey properties of random graphs, and focus on the use of the container method of Balogh–Morris–Samotij / Saxton–Thomason as a crucial tool that enables to push through proofs based on my sharp threshold criterion.

Joint work with Mathias Schacht ant Wojtek Samotij.

10:40–11:20 Doron Puder (Tel Aviv University)

Word distributions on groups. Let w be a word in the free group on k generators, and let G be a finite group. The word w induces a probability distribution on G by substituting the letters of w with k independent uniformly random elements of G and evaluating the product.

In 1896 Frobenius gave a very precise description for the distributions on finite groups given by the commutator word $[x, y] = xyx^{-1}y^{-1}$. We prove a converse of Frobenius' result and show that basically, this is the only word giving these exact distributions. I'll describe more results of this nature and state some conjectures. This is based on joint works with Ori Parzanchevski, with Liam Hanani and with Michael Magee.

11:30–12:10 Asaf Shapira (Tel Aviv University)

Two Erdős–Hajnal-type theorems in hypergraphs. The Erdős–Hajnal Theorem asserts that non-universal graphs, that is, graphs that do not contain an induced copy of some fixed graph H, have homogeneous sets of size significantly larger than one can generally expect to find in a graph. We obtain two results of this flavor in the setting of r-uniform hypergraphs.

A theorem of Rödl asserts that if an n-vertex graph is non-universal then it contains an almost homogeneous set (i.e., one with edge density either very close to 0 or 1) of size $\Omega(n)$. We prove that if a 3-uniform hypergraph is non-universal then it contains an almost homogeneous set of size $\Omega(\log n)$. An example of Rödl from 1986 shows that this bound is tight.

Let $R_r(t)$ denote the size of the largest non-universal r-graph G so that neither G nor its complement contain a complete r-partite subgraph with parts of size t. We prove an Erdős–Hajnal-type stepping-up lemma, showing how to transform a lower bound for $R_r(t)$ into a lower bound for $R_{r+1}(t)$. As an application of this lemma, we improve a bound of Conlon–Fox–Sudakov by showing that $R_3(t) \geqslant t^{\Omega(t)}$.

Joint work with M. Amir and M. Tyomkyn.

14:00–14:40 Paul Balister (University of Memphis)

The sharp threshold for making squares. Many of the fastest known algorithms for factoring large integers rely on finding subsequences of randomly generated sequences of integers whose product is a perfect square. Motivated by this, in 1994 Pomerance posed the problem of determining the threshold of the event that a random sequence of N integers, each chosen uniformly from the set $\{1, \ldots, x\}$, contains a subsequence, the product of whose elements is a perfect square. In 1996, Pomerance gave good bounds on this threshold and also conjectured that it is sharp.

In a paper published in Annals of Mathematics in 2012, Croot, Granville, Pemantle and Tetali significantly improved these bounds, and stated a conjecture as to the location of this sharp threshold. In recent work, we have confirmed this conjecture. In this talk I shall give a brief overview of some of the ideas used in the proof, which relies on techniques from number theory, combinatorics, and stochastic processes. Joint work with Béla Bollobás and Robert Morris.

14:40–15:20 Ramarathnam Venkatesan (Microsoft Research)

Rigorous analysis for a randomised number field sieve

16:20–17:00 Jonathan Lee (Microsoft Research)

Generalisations of Pomerance's problem for the number field sieve

Friday, 27 July 2018

10:50–11:30 Gábor Tardos (Rényi Institute)

On a graph coloring conjecture of Erdős and Hajnal. An old and beautiful conjecture of Erdős and Hajnal states that large chromatic graphs have large girth, large chromatic subgraphs. More precisely, for every m and k there is $n = f(m,k)$ such that every graph with chromatic number n has a subgraph of chromatic number at least m and girth at least k. Rödl proved in 1977 the girth $k = 4$ case, the rest of the conjecture is open. Following Mohar and Wu, we (who did this for Kneser graphs) prove the conjecture for special classes of graphs, including shift graphs and a family of graphs used by Burling and others as high chromatic triangle-free geometric intersection graph constructions. In the proof we find a very low threshold $f(n,k)$ for the shift graphs, but very high thresholds for Burling graphs. We show conjectures on certain 2-player combinatorial games that may actually lead to strong lower bounds on $f(n,k)$.

This is joint work with Bartosz Walczak.

11:30–12:10 Annika Heckel (Oxford)

Colouring dense random graphs and the second moment method. In a (proper vertex) coloring of a graph G, no two neighbouring vertices are coloured the same, and the chromatic number of G is the minimum number of colours where this is possible.

We study colourings of the dense random graph $G(n, pN)$ (with p constant) with constraints on the colour class sizes. In an equitable colouring, the colour class sizes may differ by at most 1. If $p < 1 - 1/e$, it turns out that the equitable chromatic number is concentrated on at most two consecutive values. This result is perhaps surprisingly sharp since, like the normal chromatic number, the equitable chromatic number is quite large in absolute value, namely of order $n/\log n$.

We say that a colouring is 2-bounded if it does not contain any colour classes of size α or $\alpha - 1$, where α is the independence number (and maximum colour class size) of G. The 2-bounded chromatic number is also concentrated on two consecutive values. For $p < 1 - 1/e$, this slightly improves the currently best known upper bound on the normal chromatic number.

Both of these results are proved through the second moment method, and we discuss more generally under which constraints the second moment method can be applied to colourings of dense random graphs.

Joint work with Konstantinos Panagiotou.

14:00–14:40 Nick Wormald (Monash University)

On the distribution of small subgraph counts in a random graph. Since the early studies of random graphs, one of the properties considered has been the distribution of subgraph counts. New results relate to copies of a strictly balanced subgraph H in moderately sparse random graphs. We obtain an asymptotic formula for the point probabilities in the distribution, i.e. the probability that there are precisely k copies, for all points except those in a 'small' upper tail. Our results apply whenever the average degrees of the random graphs are reasonably below the threshold at which each edge is included in a copy of H. The results apply to both $\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$.

Joint work with Dudley Stark.

14:40–15:20 Rafael Mendes de Oliveira (University of Toronto)

Scaling problems & applications to extremal combinatorics. Scaling problems (and their algorithms) have recently found a myriad of applications in different areas of mathematics, such as in extremal combinatorics (quantitative generalizations of Sylvester–Gallai theorem), invariant theory (null-cone problem for matrix semi-invariants), functional analysis (Brascamp–Lieb inequalities), non-commutative algebra (word problem for free-skew field) and many others.

In this talk, we will first define scaling problems and then discuss some of the structural theory behind them. After that, we will elaborate on the applications of this structural theory to problems in extremal combinatorics, in particular discussing how such scaling problems can give us quantitative generalizations of the Sylvester–Gallai theorem.

Based on joint work with Zeev Dvir, Ankit Garg, and József Solymosi.

15:40–16:20 László Babai (University of Chicago)

Symmetry versus regularity. Regularity can be thought of as a combinatorial relaxation of symmetry. Symmetry tends to have strong structural implications, often derived using heavy group theoretic machinery. Do regularity assumptions suffice for similar consequences? We review various manifestations of this meta-question, recent results, and open questions.